Stable Adaptive Control of a Class of Continuous-Flow Bioreactors

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A stable adaptive control strategy is suggested for a class of continuous-flow bioreactor processes described by Monod kinetics with two unknown parameters, one of which appears nonlinearly. Similarly, as in the case of the previously reported adaptive controllers, the parametrization of the process model, in conjunction with the adaptive exponential feeding strategy and corresponding adaptive algorithms, results in a stable system in which the convergence of the output errors to zero is guaranteed. In the former, however, two major problems are encountered: (1) both output errors were used to adjust the controller parameters, which may yield unacceptable performance of the resulting adaptive system; (2) conditions under which the process output can assume only positive values are difficult to derive. Hence, a design of a stable adaptive controller is suggested, whose parameters are adjusted using only one of the output errors and that yields acceptable performance of the control system. With this method, conditions under which the process outputs can assume only positive values can be readily derived. These conditions in turn guarantee that the control input saturation at value zero cannot occur. In this context, two adaptive controllers are suggested, such that the resulting adaptive systems are stable and the control objective is met. The adaptive controller design relies on a convenient coordinate transformation, while the proof of stability is based on suitably chosen Lyapunov functions. The performance of the adaptive system is evaluated through computer simulations.

Introduction

Optimization of continuous-flow bioreactor processes can be generally achieved by maintaining the process outputs close to their desired values over a time interval of interest. In particular, during the course of the process the cell and substrate concentrations are to be close to prespecified steady states, and the latter are chosen to optimize certain performance criteria. The corresponding control problem commonly consists of two stages: in the first stage the process that starts from some initial state is to be brought close to the desired steady state, while in the second stage the objective is to maintain the process outputs close to their desired values over large time intervals. Such a problem is rendered complex due to the highly nonlinear nature of the process, and since the control objective is to be achieved in the presence of parameter time variations and substantial unmodeled dynamics. The latter effects generally tend to deteriorate the performance of the process in a direction of lower productivity. Furthermore, the key process variables, that is, cell and substrate concentrations, are in general not measurable online. Even when full state measurement is assumed, the control problem is far from trivial due to the process complexity and parametric and structural uncertainty. Hence some type of adaptation appears to be essential in assuring acceptable process performance. Difficulties encountered when attempting to control such complex processes have been recognized by the researchers in the field, and a large number of approaches has been suggested in this context. The results pertaining to nonadaptive and adaptive control of continuous-flow bioreactor processes are, respectively, summarized in Agrawal and Lim (1984) and in Bastin and Dochain (1990). The approach from Bastin and Dochain (1990) is used to solve the adaptive control problem for a large class of bioreactor process growth models, and the design of corresponding adaptive controllers is based on an assumption that only the parameters that enter linearly in the process model are unknown. However, since in some cases the process parameters that appear nonlinearly in the growth model may be unknown and time-varying, such an approach may not give a complete solution to the control problem. Hence in Bošković (1994a,b, 1995) adaptive controller design, taking into account nonlinear parametrization of bioreactor process models, has been studied. In Bošković (1994b), a feedback-linearizing controller was used to control a class of fed-batch bioreactors, and it was shown that when two control inputs are used, the process outputs are close to desired trajectories despite the presence of unknown time-varying parameters and unmodeled dynamics. However, since the implementation of two control inputs may be a complex and expensive solution, in Bošković (1994a) an adaptive control strategy was suggested for a class of uncertain bioreactor processes in which the substrate feeding rate is the only control input. The design was based on the exponential feeding strategy that arises naturally in the context of fed-batch bioreactor control (Yamane and Shimizu, 1984), and, in the case when process parameters are known, yields a nonlinear system in which the convergence of the output errors to zero is guaranteed for a set of acceptable initial conditions. As a first step toward more complex designs, in Bošković (1994a) it was assumed that two process parameters, one of them appearing nonlinearly, are unknown, while the yield coefficient was assumed to be known. In particular, it was shown that the parametrization of the process nonlinearity, together with suitably chosen control laws and adaptive algorithms, yields a stable overall system in which the convergence of the output errors to zero is assured, despite the fact that one of the unknown process parameters enters nonlinearly into the process model. The proof of stability was based on the Lyapunov function with a suitably chosen cubic term.

However, the method from Bošković (1994a) is based on parameter adjustment laws that depend on both output errors, which may result in unacceptable transient response, particularly in cases when initial conditions in the process outputs are large. Further, the control input becomes zero when either of the process outputs (cell or substrate concentration) becomes zero, which irreversibly switches the process from continuous to batch mode of operation and the process becomes operationally uncontrollable. Since during the course of adaptation some of the process outputs can assume the value zero, conditions imposed on the free design parameters and on the transient response of the adaptive system are needed to prevent such a situation. However, when the method from Bošković (1994a) is used, very little can be said about the transient bounds of the process outputs. Nevertheless, the results from Bošković (1994a), combined with those from Bošković (1994b), serve as an important guideline for the design of bioreactor adaptive control systems in which stability and operational controllability are assured for all time. Hence, in this article we suggest a stable adaptive control strategy that results in parameter adjustment laws that depend only on the output error in substrate concentration and yields improved system performance, and, most importantly, enables us to derive sufficient conditions under which the process outputs can assume only positive values. The strategy is based on a convenient coordinate transformation, parametrization of process nonlinearities, and a suitably chosen Lyapunov function with a cubic term. We will also show that the adaptive laws can be modified so that a quadratic Lyapunov function exists for the resulting adaptive system.

The article is organized as follows: In the following section the control problem is stated, while in the third section Lyapunov's second method, as well as its application in the context of adaptive systems, are outlined. In the fourth section an idealized controller is suggested for the class of bioreactor processes. Following that, the new method suggested in this article is presented, while the simulation results are given in the sixth section. Following the list of references, an Appendix is given that contains the proof of Lemma 1 and an outline of the properties of adaptive algorithms with projection.

Problem Statement

In this article we consider a class of continuous-flow bioreactor processes described by the following model:

$$\frac{dX}{dt} = \frac{\mu_m SX}{K_s + S} - DX, \qquad X(0) = X_0, \quad (1)$$

$$\frac{dS}{dt} = -\frac{\mu_m SX}{Y(K_s + S)} + (S_F - S)D, \quad S(0) = S_0, \quad (2)$$

where X and S denote, respectively, biomass and substrate concentrations; D denotes the dilution rate, $D = F/V_R$; F denotes the feeding rate; V_R denotes the volume of the bioreactor content, S_F denotes the influent substrate concentration, μ_m is the maximum value of the growth rate; K_s is the value of S at which $\mu = \mu_m/2$; and Y denotes the yield coefficient. The influent substrate concentration S_F is constant, and the control input to the process is D.

Steady state

The steady state of the process is obtained when we set $dX/dt = 0$ and $dS/dt = 0$. Process 1 and 2 have two steady states. One is defined below:

$$\bar{S} = \frac{K_s \bar{D}}{\mu_m - \bar{D}}, \quad \bar{X} = Y(S_F - \bar{S}), \quad (3)$$

where $\bar{D} = \mu_m \bar{S}/(K_s + \bar{S})$ denotes the nominal control input. Further, $0 < \bar{S} < S_F$ and $0 < \bar{X} < YS_F$. The other steady states are $\bar{X} = 0$ and $\bar{S} = S_F$. While Eq. 3 is the desired steady state, the steady state $(0, S_F)$ corresponds to the undesired washout condition.

In this article our goal is to design a controller to keep the process close to the desired steady state (Eq. 3) when μ_m and K_s are unknown.

To make the problem under consideration analytically tractable, we assume certain prior information regarding the process. Let $p = [\mu_m \, K_s]^T$ denote the vector of unknown process parameters, and let

$$\mathbb{S}_p = \{p : p = [\mu_m \, K_s]^T, \ \mu_m \in [(\mu_m)_{\min}, (\mu_m)_{\max}], \\ K_s \in [(K_s)_{\min}, (K_s)_{\max}]\} \quad (4)$$

denote the parameter set. We make the following assumption.

Assumption 1. (i) $p \in \mathbb{S}_p$, and (ii) $X(t)$ and $S(t)$ are measurable on-line, and (iii) $0 < S_0 < \bar{S}$, and $0 < Y\bar{S} \leq X_0 < \bar{X}$.

Comment

1. One of the characteristics of bioreactor processes is that process parameters are generally not known exactly and may be time-varying. However, for a given microorganism and given substrate, the bounds on these parameters may be known. Hence such an assumption may be realistic in practice.

2. One of the main problems when controlling bioreactor processes is that key process variables are not measurable on-line. However, relatively accurate estimates of $X(t)$ can be obtained using secondary measurements, while $S(t)$ can be determined using biosensors. Further, as argued in Bošković and Narendra (1995), the adaptive control problem for bioreactor processes should be attempted in several stages of increasing complexity, and observer-based adaptive controllers should be designed only after the properties of adaptive systems resulting from the implementation of full state feedback controllers are thoroughly studied and well understood. Hence, in this article we consider the case when the full state of the process is available.

3. It is seen that S_0 is assumed to be less than $\bar{S}$. Such constraint is typical in situations when production of undesired by-products starts at substrate concentration values above $\bar{S}$, or in depollution control problems, when the substrate concentration at the outlet should be lower than $\bar{S}$ for all time.

Control Objective. Under Assumption 1, design a feedback controller such that

$$\lim_{t \to \infty} [X(t) - \bar{X}] = \lim_{t \to \infty} [S(t) - \bar{S}] = 0.$$

In other words, the objective is to design a controller that will achieve the objective in the presence of parametric uncertainty. Since it has been shown that parameter adaptive control is well suited for dealing with parametric uncertainty (Narendra and Annaswamy, 1988), in this article we suggest an adaptive controller to achieve the control objective for process 1 and 2. One of the possible approaches to the design of adaptive controllers employs the Lyapunov stability method (Hahn, 1963). Since such an approach is also used in this article, in the next section we outline this method, as well as its application in the context of the design of adaptive controllers.

Comment. While the Monod model used earlier is the simplest possible kinetics encountered in bioreactor processes, the corresponding adaptive control problem is complex due to the fact that one of the unknown parameters, that is, K_s, appears nonlinearly in the description of the process. Hence the adaptive methods developed for linear parametrizations cannot be directly applied, and a different approach is needed to deal with this type of model. Further, a successful solution of the control problem for a process described by the Monod kinetics can be an important step in the controller design in the case of more complex models encountered in bioreactor processes (such as processes governed by Contois or Haldene kinetics or by product inhibition kinetics).

Lyapunov's Second (Direct) Method

Lyapunov's second (direct) method (Hahn, 1963) is probably the most widely used technique in stability analysis of nonlinear dynamical systems. In this section we outline the main definitions and the stability theorem that represent the main trust of the method. Following that, an application of the method in the context of the design of adaptive controllers is outlined. For further details regarding this method the reader is referred to the large available body of literature on the subject.

Outline of the method

Let a system be described by the following vector differential equation:

$$\dot{x} = f(x,t), \qquad x(0) = x_0, \tag{5}$$

where $x \in \mathbb{R}^n$ denotes the state vector, and $f: \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a smooth vector function such that the solution $x(t; x_0, t_0)$ of Eq. 5 exists for all $t \geq t_0$. Further, let $f(0,t) = 0$, $\forall t \geq t_0$.

Definition 1. The equilibrium state $x_e = 0$ of system 5 is *stable* if and only if for every $\epsilon > 0$ and every $t \geq t_0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\|x_0\| \leq \delta$ implies that $\|x(t; x_0, t_0)\| \leq \epsilon$ for all $t \geq t_0$.

Definition 2. The equilibrium state $x_e = 0$ of system 5 is *attractive* if and only if for some $\rho > 0$ and for every $\Delta > 0$ and every $t_0 > 0$, there exist a fixed $T(\Delta, x_0, t_0)$ such that $\|x_0\| < \rho$ implies that $\|x(t; x_0, t_0)\| < \Delta$, for all $t \geq t_0 + T$.

Definition 3. The equilibrium state $x_e = 0$ of system 5 is *asymptotically stable* if and only if it is both stable and attractive.

Definition 4. The equilibrium state $x_e = 0$ of system 5 is *uniformly stable* if and only if δ from Definition 1 is independent of the initial time t_0.

Definition 5. The equilibrium state $x_e = 0$ of system 5 is *uniformly asymptotically stable* if and only if it is uniformly stable, and for some $\epsilon_1 > 0$ and for every $\epsilon_2 > 0$, there exist a fixed $T(\epsilon_1, \epsilon_2)$ such that $\|x_0\| < \epsilon_1$ implies that $\|x(t; x_0, t_0)\| < \epsilon_2$, for all $t \geq t_0 + T$.

Based on these definitions, other properties can also be defined, such as global stability and exponential stability. For further details the reader is referred to Hahn (1963).

The direct method of Lyapunov can be used to determine the stability of the equilibrium $x_e = 0$ of system 5 without explicitly determining the solution of the system. It involves finding a suitable scalar function $V(x,t)$ and examining its first time derivative along the motions of the system. The following theorem gives sufficient conditions for stability (asymptotic stability) of the equilibrium $x_e = 0$ of 5.

Theorem 1. The equilibrium of system 5 is uniformly stable (uniformly asymptotically stable) if there exist a scalar function $V(t,x)$ with continuous first partial derivatives with respect to x and t, such that $V(0,t) = 0$, and if the following conditions are satisfied:

1. $V(t,x)$ is *positive definite*, that is, there exists a continuous nondecreasing scalar function $\alpha(\|x\|)$ such that $\alpha(0) = 0$ and $V(x,t) \geq \alpha(\|x\|) > 0$, for all $x \neq 0$ and all $t \geq t_0$.

2. $V(t,x)$ is *decrescent*, that is, there exists a continuous nondecreasing scalar function $\beta(\|x\|)$ such that $\beta(0) = 0$, and $\beta(\|x\|) > 0$ and $V(x,t) \leq \beta(\|x\|)$, for all $x \neq 0$ and all $t \geq t_0$.

3. $V(t,x)$ is *radially unbounded*: $\lim_{\|x\| \to \infty} \alpha(\|x\|) = \infty$.
4. $V(t,x)$ is *negative semidefinite* (*negative definite*), that is,

$$\dot{V}(x,t) = \frac{\partial V}{\partial t} + (\nabla V)^T f(x,t) \leq -\gamma(\|x\|),$$

where $\gamma(\|x\|)$ is a continuous nondecreasing function such that $\gamma(0) = 0$ and $\gamma(\|x\|) \geq 0$ [$\gamma(\|x\|) > 0$] for all $x \neq 0$ and all $t > t_0$.

In the preceding theorem, part 4, ∇V denotes the gradient of V with respect to x, and the time derivative of V is evaluated along the motions of system 5.

Scalar functions $V(x,t)$ satisfying the conditions of the preceding theorem are referred to as the Lyapunov functions for system 5.

While the method just given has been extensively used in the analysis of nonlinear dynamical systems, in this section we focus on its application in the context of adaptive systems. Such an application is outlined below.

Stability analysis of adaptive systems

Probably the most exhaustive study of an application of Lyapunov's second method in the context of stability analysis of adaptive systems is given in Narendra and Annaswamy (1988). A brief outline of the approach is given below.

Let a first-order error model be given in the form:

$$\dot{e} = -\lambda e + \phi\omega(x,t), \tag{6}$$

where $\lambda > 0$, $\omega: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a smooth function of x and t such that $|x(t)| \leq k_1$, $\forall t \geq 0$, implies that $|\omega(x(t),t)| \leq k_2$, $\forall t \geq 0$, where $k_1 > 0$ and $k_2 > 0$. Further, let $e = x - x^*$, where x^*: $\mathbb{R}^+ \to \mathbb{R}$ is a bounded function, and let $\phi = \theta - \theta^*$ denote the parameter error, where θ is an adjustable parameter, and θ^* is constant.

The objective is to adjust $\theta(t)$ so that $\lim_{t \to \infty} e(t) = 0$.

We suggest the following adjustment law (adaptive algorithm):

$$\dot{\phi} = \dot{\theta} = -\gamma e \omega, \tag{7}$$

where $\gamma > 0$ denotes the adaptive gain.

We further need to verify that the equilibrium $(e, \phi) = (0,0)$ of systems 6 and 7, is stable and that the preceding objective is met. Thus we choose the following Lyapunov function candidate:

$$V(e,\phi) = \frac{1}{2}\left[e^2 + \frac{\phi^2}{\gamma}\right]. \tag{8}$$

It is clear that this function has continuous derivatives with respect to e and ϕ, and that it is positive definite, decrescent, and radially unbounded (recall the previous section). We now take the derivative of $V(e,\phi)$ along the motions of 6 and 7, which yields

$$\dot{V}(e,\phi) = -\lambda e^2 \leq 0, \tag{9}$$

for all $e \neq 0$, all $\phi \neq 0$, and all $t \geq t_0 = 0$. From the previous theorem we can now conclude that the equilibrium $(e,\phi) = (0,0)$ of the systems 6 and 7 is uniformly stable. Hence, $e \in \mathcal{L}^\infty$ and $\phi \in \mathcal{L}^\infty$, which also implies that $x \in \mathcal{L}^\infty$ and $\theta \in \mathcal{L}^\infty$, where $\mathcal{L}^\infty$ denotes a set of bounded functions. Further, upon the integration of the derivative of V we obtain

$$V(\infty) - V(0) = -\int_0^\infty \lambda e^2(\tau)\, d\tau, \tag{10}$$

which implies that $e \in \mathcal{L}^2$, that is, $e(t)$ belongs to a set of square-integrable functions. Since $x(t)$ is bounded, so is $\omega(x,t)$, and since $e(t)$ and $\phi(t)$ are bounded, we can conclude that $\dot{e} \in \mathcal{L}^\infty$. Now, since $e \in \mathcal{L}^2$ and $\dot{e} \in \mathcal{L}^\infty$, it follows from Barbalaat's lemma (Narendra and Annaswamy, 1988) that $\lim_{t \to \infty} e(t) = 0$.

Idealized Controller

In this article our objective is to use an approach based on the Lyapunov's second method, as applied in the context of adaptive systems, to design an adaptive controller for the process 1 and 2. Before proceeding to the design stage, we first show that the solution to the adaptive control problem exists. Similar to Bošković (1994a), in this article the basis for the adaptive controller design is the exponential feeding strategy (Yamane and Shimizu, 1984) of the form:

$$D = \frac{\mu_m S X}{(K_s + S)\bar{X}}. \tag{11}$$

After substituting expression 11 into Eqs. 1 and 2 we obtain

$$\dot{e}_1 = -De_1, \qquad \dot{e}_2 = -De_2, \tag{12}$$

where $e_1 = X - \bar{X}$ and $e_2 = S - \bar{S}$. This system is expressed in terms of the output errors e_1 and e_2, and is referred to as the error model. We next analyze the stability of the equilibrium $e_1 = 0$, $e_2 = 0$ of this error model using Lyapunov's second method. We choose the Lyapunov function candidate in the form:

$$V(e_1, e_2) = \frac{1}{2}[e_1^2 + e_2^2]. \tag{13}$$

Taking its derivative along the motions of the error model given earlier, we obtain

$$\dot{V}(e_1, e_2) = -D[e_1^2 + e_2^2] < 0,$$

for all nonzero values of e_1 and e_2, and for all time, provided $X_0 > 0$ and $S_0 > 0$. It follows that the equilibrium $(e_1, e_2) = (0,0)$ of system 12 is uniformly asymptotically stable (recall the section on the method outline) and, hence, $\lim_{t \to \infty} e_1(t) = \lim_{t \to \infty} e_2(t) = 0$. We can conclude that, in the case when all process parameters are known, the control objective is met using the exponential feeding strategy (Eq. 11). Our objective is to use such a control law as a basis for an adaptive controller in the case when process parameters are unknown. Such a controller is considered next.

Comment. As seen from the preceding analysis, a necessary condition for the stability proof of go through is that X_0 and S_0 are strictly positive. This implies that D, as described by Eq. 11, is also strictly positive at $t=0$. It is only after this is established that we can verify the stability of the system. Due to the structure of the error model 12, it is clear that $S_0>0$ and $X_0>0$, implying $D(0)>0$, will also imply that the output errors will converge to zero, so that the conditions $X(t)\geq X_0>0$ and $S(t)\geq S_0>0$ are satisfied for all $t\geq 0$, which in turn implies that $D(t)\geq D_0>0$ for all time. However, as we will see later, when the error equations contain the terms in parameter errors as well, such ranges of values for $X(t)$ and $S(t)$ cannot be readily established. Hence separate analysis is needed in such cases, as we emphasize in the following section.

Stable Adaptive Controller

One solution to the adaptive control problem for process 1 and 2 was given in Bošković (1994a). While the approach reported in that paper yields a stable adaptive system in which the convergence of the output errors is guaranteed, the method has several shortcomings that prevent the results to be readily extended to more complex cases: (1) both output errors are used in the adaptive algorithms, which may lead to unacceptable performance of the system, particularly in cases when initial errors differ in several orders of magnitude, and (2) the stability of the system can be demonstrated only if the control input is assumed to be strictly positive for all time; on the other hand, conditions under which this will be true are very difficult to derive since the Euclidian norm of the output error vector is involved in analysis. Since these properties limit substantially the applicability of the method, in this article we suggest an approach that yields an adaptive system with the following properties: (1) the stability and convergence properties of the adaptive system are retained; (2) only e_2 is used to adjust the parameters; and (3) an efficient analysis of the transient response of the system is possible, which in turn aids in the control input saturation analysis.

The method suggested in this article to solve the adaptive control problem for process 1 and 2 is based on the following coordinate transformation (D'Ans et al., 1974):

$$z=X-Y(S_F-S). \tag{14}$$

After taking the first derivative of the preceding expression, from 1 and 2 we obtain:

$$\dot{z}=-Dz, \tag{15}$$

which implies that $\lim_{t\to\infty}z(t)=0$, provided $D(t)\geq D_{\min}>0$ for all time. The former condition also implies that

$$\lim_{t\to\infty}\{X(t)-Y[S_F-S(t)]\}=\lim_{t\to\infty}[e_1(t)+Ye_2(t)]=0.$$

Hence, if the adaptive controller is designed to assure that $\lim_{t\to\infty}e_2(t)=0$, $X(t)$ will also asymptotically reach the desired steady state $\bar{X}$.

As shown in Bošković (1994a,b), the design of stable adaptive laws depends on the parametrization of the process nonlinearities. Using a similar approach we further parametrize Eq. 2 in the form

$$\frac{dS}{dt}=-\frac{\alpha_1 SX}{Y(1+\alpha_2 S)}+(S_F-S)D, \tag{16}$$

where $\alpha_1=\mu_m/K_s$ and $\alpha_2=1/K_s$. Let $\alpha=[\alpha_1\ \alpha_2]^T$. Since μ_m and K_s are unknown, so are α_1 and α_2. Based on the definition of set $\mathbb{S}_p$ (Assumption 1), we can also define the set

$$\mathbb{S}_\alpha=\left\{\alpha:\alpha=[\alpha_1\ \alpha_2]^T,\ 0<(\alpha_j)_{\min}\leq\alpha_j\leq(\alpha)_{\max},\ j=1,2\right\}, \tag{17}$$

where $(\alpha_1)_{\min}=(\mu_m)_{\min}/(K_s)_{\max}$, $(\alpha_1)_{\max}=(\mu_m)_{\max}/(K_s)_{\min}$, $(\alpha_2)_{\min}=1/(K_s)_{\max}$, and $(\alpha_2)_{\max}=1/(K_s)_{\min}$. It follows that $\alpha\in\mathbb{S}_\alpha$.

Since α_1 and α_2 are unknown, in this case we choose the control law in the form

$$D=\frac{\hat{\alpha}_1 SX}{(1+\hat{\alpha}_2 S)\bar{X}}, \tag{18}$$

where $\hat{\alpha}_1$ and $\hat{\alpha}_2$ denote, respectively, the estimates of α_1 and α_2.

Now the objective is to adjust the controller parameters $\hat{\alpha}_1$ and $\hat{\alpha}_2$ so that the overall system is stable and the output errors e_1 and e_2 converge to zero. The first step in our design is to derive the error model of the system. We now substitute Eq. 18 into Eq. 2, and, after some straightforward algebraic manipulations, obtain:

$$\dot{e}_2=-De_2+\frac{SX\phi_1}{(1+\hat{\alpha}_2 S)Y}-\frac{\alpha_1 S^2X\phi_2}{(1+\alpha_2 S)(1+\hat{\alpha}_2 S)Y}, \tag{19}$$

where $\phi_1=\hat{\alpha}_1-\alpha_1$ and $\phi_2=\hat{\alpha}_2-\alpha_2$ denote the parameter errors.

When the coordinate transformation 14 is used, we can choose z and e_2 as the state variables of the system. Hence the error model in this case consists of Eqs. 15 and 19. To illustrate the application of Lyapunov's stability method in the context of adaptive control design for systems 1 and 2, as well as the difficulties encountered since K_s appears nonlinearly in the process model, we will consider two cases: (1) case when only μ_m is unknown, and (2) case when both μ_m and K_s are unknown. In the former case it follows that $\alpha_2=1/K_s$ is also known. Hence α_2 rather than $\hat{\alpha}_2$ can be used in the control law (Eq. 18), so that the second term on the r.h.s. of expression 19 is zero. Since in this case the state variables of the system are z, e_2, and ϕ_1, we choose the following quadratic Lyapuov function candidate:

$$V_1(z,e_2,\phi_1)=\frac{1}{2}\left[z^2+e_2^2+\frac{\phi_1^2}{2\gamma_1}\right], \tag{20}$$

where $\gamma_1>0$. It is seen that this function is positive definite, decrescent, and radially unbounded (recall the section outline of the method). After taking its first derivative along the motions of Eqs. 15 and 19, we obtain:

$$\dot{V}_1(z,e_2,\phi_1) = -D[z^2+e_2^2] + \frac{e_2 SX\phi_1}{(1+\hat{\alpha}_2 S)Y} + \frac{\phi_1\dot{\phi}_1}{\gamma}. \quad (21)$$

Now the objective is to find an adjustment law for $\hat{\alpha}_1$ so that the preceding derivative is (at least) negative semidefinite. It is seen that this can be achieved by canceling out the last two terms on the r.h.s. of Eq. 21. Keeping in mind that $\dot{\hat{\alpha}}_1 = \dot{\phi}_1$ (since α_1 is constant), we further choose

$$\dot{\phi} = \dot{\hat{\alpha}}_1 = -\frac{\gamma_1 e_2 SX}{(1+\hat{\alpha}_2 S)Y}, \quad (22)$$

where γ_1 is referred to as the adaptive gain. Hence, $\dot{V}_1(z,e_2,\phi_1) = -D[z^2+e_2^2] \le 0$ for all nonzero values of z, e_2, and ϕ_1, and for all $t \ge 0$, provided $D(t) > 0$ for all time. In such a case we can conclude that z, e_2, and ϕ_1 are bounded, and, using the standard arguments from Narendra and Annaswamy (1988) (see the section on stability analysis), it follows that $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} e_2(t) = 0$. The former condition also implies that $\lim_{t\to\infty} e_1(t) = 0$, so that the control objective is met.

The adaptive algorithm 22 is of the gradient type (Narendra and Annaswamy, 1988) and is designed for the ideal case, that is, for the case when there are no disturbances or unmodeled dynamics present. However, it cannot guarantee that all the signals will be bounded in the presence of various disturbances, and some modifications are needed to achieve the robustness of the system (Narendra and Annaswamy, 1988). One possible way of achieving this is to use the adaptive algorithms with projection (Narendra and Annaswamy, 1988; Appendix), whose main feature is that the parameter estimates are kept within known bounds for all time, even while assuring stability and robustness of the system. Since in our case the bounds on the process parameters are known, we will adjust the controller parameters using this type of adaptive algorithms. Hence we choose:

$$\dot{\phi}_1 = \dot{\hat{\alpha}}_1 = \mathrm{Proj}_{[(\alpha_1)_{\min},(\alpha_1)_{\max}]}\left\{-\frac{\gamma_1 e_2 SX}{(1+\hat{\alpha}_2 S)Y}\right\}, \quad (23)$$

where the function $\mathrm{Proj}_{[(\alpha_1)_{\min},(\alpha_1)_{\max}]}\{\cdot\}$ denotes the projection operator whose role is to assure that the estimate $\hat{\alpha}_1(t)$ belongs to the interval $[(\alpha_1)_{\min}, (\alpha_1)_{\max}]$ for all time. As shown in the Appendix, when such an operator is used, the following inequality:

$$\phi_1\dot{\phi}_1 \le -\frac{\gamma_1 e_2 SX\phi_1}{(1+\hat{\alpha}_2 S)Y}, \quad (24)$$

is satisfied for all values of arguments. Hence it can be readily verified that the stability properties of the system are retained with this type of adaptive algorithm.

The relative ease with which the preceding adaptive algorithms were designed in the case when K_s is known comes from the fact that μ_m appears linearly in the model (Eqs. 1 and 2), which results in the signal multiplying ϕ_1 in expression 19 being measurable. In the case when K_s is unknown, so is α_2, and the problem becomes more complex since the signal multiplying ϕ_2 in Eq. 19 is not measurable due to the presence of unknown α_2. Now a question arises as to which signal should be used to adjust $\hat{\alpha}_2$ to yield a stable system. One possibility is to use the so-called "certainty-equivalence adaptive algorithms," suggested by Bošković (1994b), such that in the adaptive algorithm the unknown α_2 is substituted by its estimate $\hat{\alpha}_2$. Hence we choose

$$\dot{\phi}_2 = \dot{\hat{\alpha}}_2 = \mathrm{Proj}_{[(\alpha_2)_{\min},(\alpha_2)_{\max}]}\left\{\frac{\gamma_2 e_2 S^2 X}{(1+\hat{\alpha}_2 S)^2 Y}\right\}, \quad (25)$$

where $\gamma_2 > 0$ denotes the adaptive gain. It also follows that the inequality

$$\phi_2\dot{\phi}_2 \le \frac{\gamma_2 e_2 S^2 X\phi_2}{(1+\hat{\alpha}_2 S)^2 Y}, \quad (26)$$

is satisfied for all values of arguments (Appendix). The preceding adaptive algorithm is chosen to be of the projection type not only because of the robustness considerations, but also to prevent the division by zero in the expression 18.

Now a question arises whether the preceding adaptive algorithms result in a stable system. Since quadratic Lyapunov functions are the most commonly used in stability analysis of adaptive system, such a candidate function is used in this case as well. Hence we choose

$$V_2(z,e_2,\phi_1,\phi_2) = \frac{1}{2}\left[z^2+e_2^2+\frac{\phi_1^2}{2}+\alpha_1\frac{\phi_2^2}{\gamma_2}\right], \quad (27)$$

and evaluate its derivative along the motions of the system (Eqs. 15, 19, 23 and 25) to obtain

$$\dot{V}_2(z,e^2,\phi_1,\phi_2) \le -D[z^2+e_2^2] + \frac{\alpha_1 e_2 S^3 X\phi_2^2}{(1+\alpha_2 S)(1+\hat{\alpha}_2 S)^2}.$$

Since the last term on the r.h.s of the preceding expression is sign-indefinite and cannot be shown to be bounded for all values of X, we can conclude that the quadratic Lyapunov function candidate is not well suited for the stability analysis in the case when the adaptive algorithm 25 is used to adjust $\hat{\alpha}_2$. A solution to this problem was given in Bošković (1994a) in the case when both e_1 and e_2 were used to adjust the parameters. In this article we adopt a similar approach in the context of the error model (Eqs. 15 and 19). Such an approach is given below.

Since the Lyapunov function candidate (Eq. 27) cannot be used to demonstrate the stability of the preceding system, we modify it by including a cubic term in ϕ_2,

$$V_3(z,e_2,\phi_1,\phi_2) = \frac{1}{2}\left[z^2+e_2^2+\frac{\phi_1^2}{2}+\alpha_1\frac{\phi_2^2}{\gamma_2}\right] + \alpha_1\frac{c_1}{c_2}\cdot\frac{\phi_2^3}{3\gamma_2}, \quad (28)$$

where $c_1 > 0$ and $c_2 > 0$ are to be found to assure that V_3 is positive definite, decrescent, and radially unbounded, and that its first derivative along the motions of the system is negative semidefinite. We further have

$$\dot{V}_3(z,e_2,\phi_1,\phi_2) \leq -D[z^2+e_2^2] + \alpha_1 \xi(e_2,\phi_2,S,X), \quad (29)$$

where

$$\xi(e_2,\phi_2,S,X) = -\frac{e_2\phi_2 S^2 X}{1+\hat{\alpha}_2 S}\left[\frac{1}{1+\alpha_2 S} - \frac{c_1+c_2\phi_2}{c_2(1+\hat{\alpha}_2 S)}\right],$$

is referred to as the indicator function. Now the idea is to find c_1 and c_2 such that ξ is negative semidefinite. If we choose: $c_1 = \bar{S}$ and $c_2 = 1+\alpha_2\bar{S}$, we obtain

$$\xi(e_2,\phi_2,S,X) = -\frac{e_2\phi_2 S^2 X}{1+\hat{\alpha}_2 S}\left[\frac{1}{1+\alpha_2 S} - \frac{1+\hat{\alpha}_2\bar{S}}{(1+\alpha_2\bar{S})(1+\hat{\alpha}_2 S)}\right] \quad (30)$$

and

$$\xi(e_2,\phi_2,S,X) = -\frac{e_2^2\phi_2^2 S^2 X}{(1+\alpha_2 S)(1+\alpha_2\bar{S})(1+\hat{\alpha}_2 S)} \leq 0$$

for all values of arguments.

Hence, $\dot{V}_3(z,e_2,\phi_1,\phi_2) \leq 0$ for all nonzero values of arguments and for all time, so that, if $D(t) > 0$ for all time, using the standard arguments outlined in the section on stability analysis we can readily demonstrate that $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} e_2(t) = 0$, and that, in addition, $\lim_{t\to\infty} e_1(t) = 0$.

Comment. Since $\hat{\alpha}_2$ belongs to the interval $[(\alpha_2)_{\min}, (\alpha_2)_{\max}]$ for all time due to the properties of the projection-type adaptive algorithm 25, we can readily show that $V_3(z,e_2,\phi_1,\phi_2)$ from Eq. 28, with $c_1 = \bar{S}$ and $c_2 = 1+\alpha_2\bar{S}$, is positive definite, decrescent, and radially unbounded.

Even though $V_3(z,e_2,\phi_1,\phi_2)$ satisfies all conditions of Theorem 1, it may not be best suited for further analysis due to the presence of the cubic term in ϕ_2. Hence, our objective is to modify the adaptive law for adjusting ϕ_2 so that a quadratic Lyapunov function exists for the resulting adaptive system. The motivation as to how to modify the adaptive law 25 comes from expression 30. With the objective of realizing the same *function* ξ *in the* derivative of a Lyapunov function, the following adaptive law and the Lyapunov function are found:

$$\dot{\phi}_2 = \dot{\hat{\alpha}}_2 = \mathrm{Proj}_{[(\alpha_2)_{\min},(\alpha_2)_{\max}]}\left\{\frac{\gamma_2[1+\hat{\alpha}_2\bar{S}]e_2 S^2 X}{(1+\hat{\alpha}_2 S)^2 Y}\right\}, \quad (31)$$

and

$$V_4(z,e_2,\phi_1,\phi_2) = \frac{1}{2}\left[z^2+e_2^2+\frac{\phi_1^2}{2}+\frac{\alpha_1}{1+\alpha_2\bar{S}}\cdot\frac{\phi_2^2}{\gamma_2}\right]. \quad (32)$$

It is seen that the function just given is quadratic in all system states. After taking its derivative along the motions of the system (Eqs. 15, 19, 23 and 31), we obtain

$$\dot{V}_4(z,e_2,\phi_1,\phi_2) \leq -D[z^2+e_2^2] + \alpha_1\xi(e_2,\phi_2,S,X), \quad (33)$$

where ξ is defined by expression 30. It is seen that by introducing the term $(1+\hat{\alpha}_2\bar{S})$ in the adaptive algorithm 31, it was possible to come up with a quadratic Lyapunov function 32 for the system, and that this results in obtaining the same indicator function in the expression 29 as in the case of the adjustment law (Eq. 25) and the Lyapunov function (Eq. 28).

Comment. While the preceding adaptive algorithms assure the asymptotic convergence of the output errors to zero, they cannot assure that the parameter errors will converge to zero as well. The reason for this is that the desired values of the process outputs are constant and thus not "persistently exciting" (Narendra and Annaswamy, 1988). Loosely speaking, since the process has two unknown parameters, the desired values of the process outputs are not sufficiently "rich" to excite the whole parameter space and force the parameter errors to zero. In our case, however, the objective is to force the output errors rather than the parameter errors to zero, and such an objective is achieved using adaptive laws 23 and 25 or 31.

As seen in all cases just studied, the stability of the system was proved under the assumption that the condition

$$X(t) > 0, \qquad S(t) > 0, \quad (34)$$

is satisfied for all time, which in turn guarantees that the control input saturation at value zero cannot occur. However, since we do not know *a priori* that such a condition will be satisfied, we will further derive conditions under which the process states are guaranteed to assume strictly positive values for all time.

Let us, for simplicity, assume that γ_1 and γ_2 from Eqs. 23 and 31 have the same values, that is, $\gamma_1 = \gamma_2 = \gamma$. We further consider the following lemma.

Lemma 1. If the following condition is satisfied:

$$\gamma \geq \frac{c}{[S_0 - \epsilon_S]^2}, \quad (35)$$

where

$$c = [(\alpha_1)_{\max} - (\alpha_1)_{\min}]^2 + \frac{(\alpha_1)_{\max}[(\alpha_1)_{\max} - (\alpha_1)_{\min}]^2}{1+(\alpha_2)_{\min}\bar{S}}, \quad (36)$$

and $0 < \epsilon_S < S_0$, then the conditions

$$X(t) \geq Y\epsilon_S > 0, \quad S(t) \geq \epsilon_S > 0, \quad (37)$$

are satisfied for all $t \geq 0$.

Proof. The proof is given in the Appendix.

Comment

1. While the preceding lemma is proved for a special case when $\gamma_1 = \gamma_2$, very similar conditions can be derived in the case when $\gamma_1 \neq \gamma_2$.

2. The preceding condition is only sufficient, and thus restrictive. Loosely speaking, the lemma indicates that condition 37 will be satisfied if γ_1 and γ_2 are sufficiently large. The importance of this result stems from the fact that this condition guarantees that $D(t) \geq \mu_m \epsilon_S^2 Y/[(K_s + \epsilon_S)\bar{X}] > 0$ for all time, that is, that the control input cannot saturate at $D = 0$.

Computer Simulations

In this section the method studied in the previous section will be evaluated through computer simulations. The case study involves the following process model:

$$\dot{X} = \frac{0.35SX}{0.25+S} - DX, \qquad X(0) = 10, \tag{38}$$

$$\dot{S} = -\frac{0.70SX}{0.25+S} + (200-S)D, \qquad S(0) = 0.01. \tag{39}$$

We assume that the process parameter vector $p = [0.35\ 0.25]^T$ is unknown to the designer, and that the following prior information is assumed to be available:

$$p \in \mathcal{S}_p = \{p : p = [\mu_m K_s]^T, 0.1 \leq \mu_m \leq 0.5, 0.1 \leq K_s \leq 1\}. \tag{40}$$

Based on the parametrization from the preceding section we also introduce the vector $\alpha = [\alpha_1\ \alpha_2]^T$, whose elements are $\alpha_1 = \mu_m/K_s$ and $\alpha_2 = 1/K_s$, so that

$$\alpha \in \mathcal{S}_\alpha = \{\alpha : \alpha = [\alpha_1\ \alpha_2]^T, 0.1 \leq \alpha_1 \leq 5, 1 \leq \alpha_2 \leq 10\}. \tag{41}$$

We will assume that the desired steady-state for S is $\bar{S} = 0.28$. Hence from Eq. 3 we have that $\bar{X} = 99.86$.

As mentioned earlier, in this article we will assume that $S(t)$ cannot be allowed to exceed $\bar{S}$ for extended periods of time. This is a realistic requirement encountered in the following situations:

1. When the process is substrate inhibited and the inhibition effect on the cell growth rate becomes substantial above $\bar{S}$.
2. When formation of undesired products starts at concentrations above $\bar{S}$ (e.g., ethanol formation in the case of baker's yeast (Takamatsu et al., 1985).
3. In the depollution control problems, when the fermentation broth having substrate concentration higher than $\bar{S}$ cannot be released in nature.

Hence, the process is operated in fed-batch mode until some prespecified volume is reached, when the pump at the outlet, having the same flow rate as that at the inlet, is activated. Since in both cases the structure of the resulting control laws and error models is the same, we will consider the process assuming that it is operated in the continuous mode for all time.

Controller design

In all cases considered in this section the initial conditions of the parameter estimates are chosen as

$$\hat{\alpha}_1(0) = \frac{(\alpha_1)_{\max} + (\alpha_1)_{\min}}{2} = 2.55,$$

$$\hat{\alpha}_2(0) = \frac{(\alpha_2)_{\max} + (\alpha_2)_{\min}}{2} = 5.5.$$

The process is simulated over the interval [0, 100] hours using Euler's method employing the step size 0.001. In the case of adaptive controllers, the adaptive gains were in all cases chosen as $\gamma_1 = 10$, and $\gamma_2 = 100$, using trial and error.

Fixed controller

We first show that the control problem cannot be solved using a fixed controller obtained when the adaptive gains were set to zero and the initial values of the controller parameters were chosen as in Eq. 42. In such a case, the response of the system is given in Figure 1, in which $X_d(t)$ denotes the response of the system when true values of parameters were used in the control law 18. The responses of both $X(t)$ and $S(t)$ are not acceptable and that their steady-state values are far from the desired ones defined earlier (i.e., $\bar{S} = 0.28$ and $\bar{X} = 99.86$). These responses can be improved using linear techniques by, for instance, adding an integral term to the controller 18. However, when a linear controller is used to control processes 1 and 2, very little can be said about the stability and transient response of the corresponding closed-loop system. Hence we further focus on response achieved using the adaptive controller 18, 23 and 31 whose stability and transient properties are established in this article.

Adaptive controller

The response of the system in the case when the adaptive controller 18, 23 and 31 was used is shown in Figure 2 in the case when process parameters are constant. It is seen that the overall performance is good and that $e_2(t)$ after an initial transient quickly converges to zero. The adaptive system is also simulated in the case when $X(t)$, $S(t)$, and S_F were corrupted by zero mean noise, Figure 3. The signal-to-noise ratio was chosen to be 10 for the steady-state values of $X(t)$

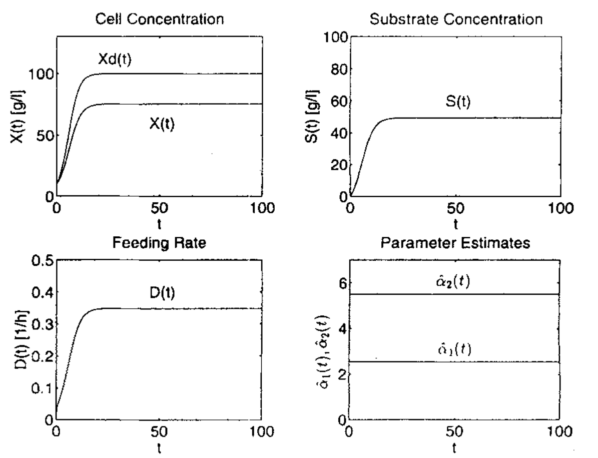

Figure 1. Response of the system with the fixed controller.

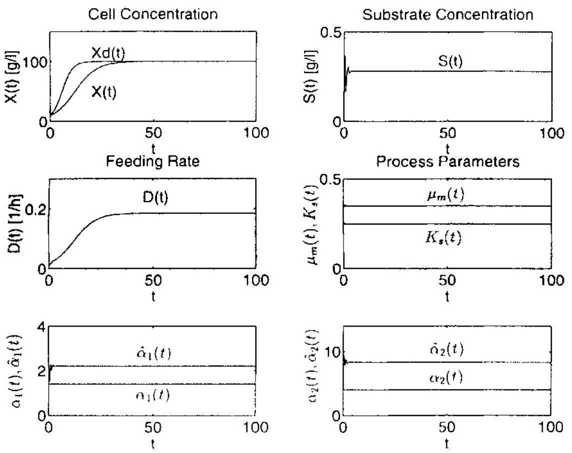

Figure 2. Response of the system with the adaptive controller: case with no noise.

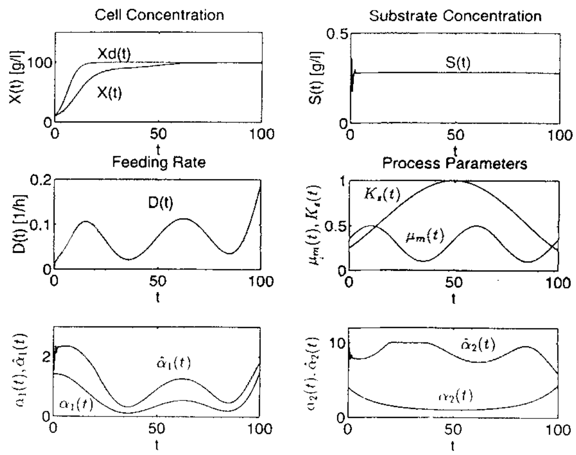

Figure 4. Response of the system with the adaptive controller when process parameters vary with time: case with no noise.

and $S(t)$, and for S_F itself. Hence the corresponding variances were chosen as 10, 0.028, and 20, respectively. It is seen that in both cases from Figures 2 and 3, as indicated by theory, the controller parameters do not converge to their true values.

The system is further simulated in the case when μ_m and K_s vary with time as shown in Figures 4 and 5; also in cases with and without noise. It is seen that the response in $S(t)$ is good, while due to the parameter time variations, response in $X(t)$ is somewhat slower than in the cases from Figures 2 and 3. Since μ_m and K_s vary with time, so do α_1 and α_2. The latter and their estimates are shown in the lower portion of Figures 4 and 5.

Also, in all simulations the actual response of the system is compared to the desired response $X_d(t)$ obtained when true values of parameters are used in the control law. It is seen that in all cases the actual response is slower than that of $X_d(t)$, which is due to the presence of parametric uncertainty of the process. Hence, even though $X(t)$ reaches the desired steady state, the settling time is larger than that in the case of $X_d(t)$.

From the preceding simulations the following conclusions can be made:

1. In the case of no noise, the method suggested in this article yields an acceptable performance of the adaptive system.
2. The method is also robust to large-parameter time variations and to large noise in input and output variables.
3. The choice of the values of adaptive gains prevents the control input saturation at value zero.

Conclusion

In this article the adaptive exponential feeding strategy was designed for a class of nonlinearly parametrized models arising in continuous-flow bioreactor processes. Under the assumption that the yield coefficient is known, the system consisting of the controlled process and adaptive laws is shown to be stable and the output errors are guaranteed to converge to zero in the case when only $e_2(t)$ was used to adjust

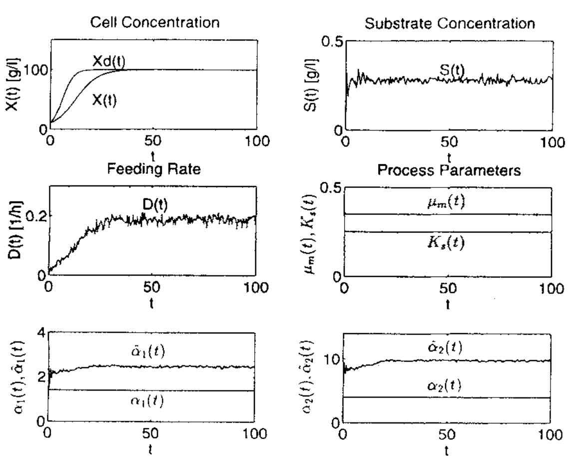

Figure 3. Response of the system with the adaptive controller: case when noise is present in $X(t)$, $S(t)$, and S_F.

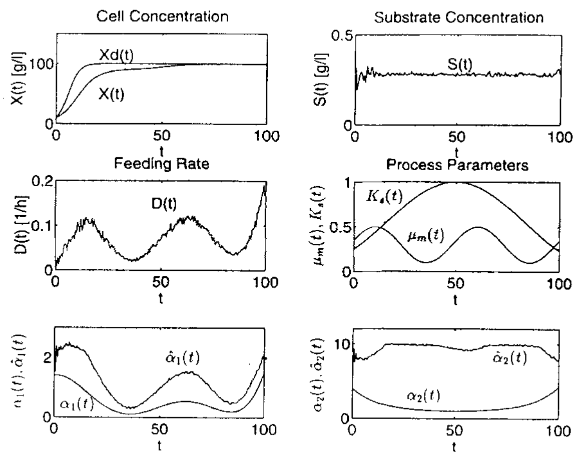

Figure 5. Response of the system with the adaptive controller when process parameters vary with time: case when noise is present in $X(t)$, $S(t)$, and S_F.

the parameters. Two cases of adaptive algorithms were considered, and in both cases the proof of stability of the overall system is carried out using a suitably chosen Lyapunov function. Further, the adaptive scheme is guaranteed to avoid control input saturation at value zero under the conditions imposed on the adaptive gains. Through a simulation study the method was shown to yield acceptable performance of the closed-loop system in the presence of large noise and parameter time variations. The results from this article represent one of the stages of increasing complexity toward a solution to the adaptive control problem in the case when all process parameters are unknown, more complex growth models are encountered, unmodeled dynamics are present, and only one of the process outputs is measurable. In the latter case, an observer-based adaptive controller is called for. Development of one such controller is currently in progress.

Notation

c, c_1, c_2 = constants
$\bar{S}$ = steady-state value of substrate concentration
$\bar{X}$ = steady-state value of cell concentration
α = parameter vector, $\alpha = [\alpha_1 \ \alpha_2]^T$
α_1, α_2 = parameters of the parametrized process model
$\hat{\alpha}_1, \hat{\alpha}_2$ = parameter estimates
ϵ_S = constant, $0 < \epsilon_S < S_0$
$\phi_1 \ \phi_2$ = parameter errors

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Appendix

Proof of Lemma 1

From expression 32 we have

$$V_4(z, e_2, \phi_1, \phi_2) \le \frac{1}{2}\left[z^2 + e_2^2 + \frac{c}{\gamma}\right],$$

where c is defined by Eq. 36. From the preceding expression and Eq. 33 we further have

$$\dot{V}_4 \le -2D\left[V_3 - \frac{c}{2\gamma}\right]. \tag{A1}$$

Using the comparison principle approach (Yoshizawa, 1966), and since $V_4(\cdot) > 0$ for all nonzero arguments, we further have

$$V_4(t) \le \left[V_4(0) - \frac{c}{2\gamma}\right]\exp\left\{-\int_0^t D(\tau)\,d\tau\right\} + \frac{c}{2\gamma}. \tag{A2}$$

Further, since from expression 32 at $t = 0$ we have

$$V_4[z(0), e_2(0), \phi_1(0), \phi_2(0)] \le \frac{1}{2}\left[z^2(0) + e_2^2(0) + \frac{c}{\gamma}\right],$$

and since

$$V_3(z, e_2, \phi_1, \phi_2) \ge \frac{1}{2}[z^2 + e_2^2]$$

for all values of the arguments, we now have

$$z^2(t) + e_2^2(t) \le [z^2(0) + e_2^2(0)]\exp\left\{-2\int_0^t D(\tau)\,d\tau\right\} + \frac{c}{\gamma}.$$

Since from Eq. 15 we have

$$z(t) = z(0)\exp\left\{-\int_0^t D(\tau)\,d\tau\right\},$$

from the two previous expressions we obtain

$$|e_2(t)| \le |e_2(0)|\exp\left\{-\int_0^t D(\tau)\,d\tau\right\} + \sqrt{\frac{c}{\gamma}}. \tag{A3}$$

The critical case is when $S(t') = 0$ for some t', which can occur when $S(t) < \bar{S}$. Hence we consider the latter case and rewrite Expression 44 in the form

$$S(t) \ge \bar{S} - [\bar{S} - S_0]\exp\left\{-\int_0^t D(\tau)\,d\tau\right\} - \sqrt{\frac{c}{\gamma}} \ge S_0 - \sqrt{\frac{c}{\gamma}}, \tag{A4}$$

where the last inequality is based on the fact that $S_0 < \bar{S}$ [recall Assumption 1 (iii)]. It now follows that the condition $S(t) \ge \epsilon_S$ (where ϵ_S is defined in the statement of the lemma) will be satisfied for all time provided

$$\gamma \ge \frac{c}{[S_0 - \epsilon_S]^2}. \tag{A5}$$

We further consider the condition on $X(t)$. Since $z = e_1 + Ye_2$, from Eq. A3 we now have

$$e_1 + Ye_2 = [e_1(0) + Ye_2(0)] \exp\left\{-\int_0^t D(\tau)\, d\tau\right\} \quad \text{(A6)}$$

Again, the critical case is when $X(t'') = 0$ for some t'', which can occur when $X(t) < \bar{X}$. In the latter case, the preceding expression is rewritten in the form (recall the definition of e_1 and e_2):

$$X(t) - \bar{X} + Y[S(t) - \bar{S}] \geq \{X(0) - \bar{X} + Y[S(0) - S]\} \times \exp\left\{-\int_0^t D(\tau)\, d\tau\right\} \quad \text{(A7)}$$

for all time, and

$$\bar{X} - X(t) + Y[\bar{S} - S(t)] \leq \{\bar{X} - X(0) + Y[\bar{S} - S(0)]\} \exp\left\{-\int_0^t D(\tau)\, d\tau\right\} \leq \{\bar{X} - X(0) + Y[\bar{S} - S(0)]\},$$

for all $t \geq 0$, from where we have

$$X(t) \geq -YS(t) + X_0 + YS_0.$$

Since from Eq. A3 and the condition A5 we can conclude that $S(t) \leq 2S_0 - \epsilon_S$ for all time, we now have

$$X(t) \geq X_0 - YS_0 + Y\epsilon_S \geq Y\epsilon_S > 0,$$

for all time [recall Assumption 1 (iii)], which completes the proof. □

Projection-type adaptive algorithms

Properties of such adaptive algorithms will be illustrated in the example of a simple error model from the section on stability analysis. Let $\theta^* \in [\theta_{\min}, \theta_{\max}]$. The adaptive algorithm takes the form:

$$\dot{\phi} = \dot{\theta} = \mathrm{Proj}_{[\theta_{\min}, \theta_{\max}]}\{-\gamma e \omega\},$$

where the projection operator is defined as follows:

$$\mathrm{Proj}_{[\theta_{\min}, \theta_{\max}]}\{-\gamma e\omega\} = \begin{cases} -\gamma e\omega, & \text{if } \theta(t) = \theta_{\max} \text{ and } e\omega > 0, \\ 0, & \text{if } \theta(t) = \theta_{\max} \text{ and } e\omega \leq 0, \\ -\gamma e\omega, & \text{if } \theta_{\min} < \theta(t) < \theta_{\max}, \\ 0, & \text{if } \theta(t) = \theta_{\min} \text{ and } e\omega \geq 0, \\ -\gamma e\omega, & \text{if } \theta(t) = \theta_{\min} \text{ and } e\omega < 0. \end{cases}$$

Our objective is to show that in all of the cases $\phi\dot{\phi} \leq -\gamma e\omega\phi$, that is, that $\phi\dot{\phi} + \gamma e\omega\phi \leq 0$. We further consider each individual case:

(i) $\theta(t) = \theta_{\max}$. When $e\omega > 0$, we have that $\dot{\phi} = -\gamma e\omega$, and $\phi\dot{\phi} = -\gamma e\omega\phi$. Since $\phi = \theta - \theta^*$ and $\theta^* \in [\theta_{\min}, \theta_{\max}]$, in this case $\phi = \theta_{\max} - \theta^* \geq 0$. When $e\omega \leq 0$, $\dot{\phi} = 0$, and we have $\phi\dot{\phi} + \gamma e\omega\phi = \gamma e\omega\phi \leq 0$.

(ii) $\theta_{\min} < \theta(t) < \theta_{\max}$. In this case $\phi\dot{\phi} = -\gamma e\omega\phi$.

(iii) $\theta(t) = \theta_{\min}$. For $e\omega < 0$, we have that $\dot{\phi} = -\gamma e\omega$, and $\phi\dot{\phi} = -\gamma e\omega\phi$. Since in this case $\phi = \theta_{\min} - \theta^* \leq 0$, for $e\omega \geq 0$, we have $\dot{\phi} = 0$, and $\phi\dot{\phi} + \gamma e\omega\phi = \gamma e\omega\phi \leq 0$.

It follows that the adaptive algorithms with projection assure that the condition $\phi\dot{\phi} \leq -\gamma e\omega\phi$ is satisfied for all values of arguments.

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